INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 4, 283–290 (1984)

A NATURAL INTERPOLATION FORMULA FOR PRANDTL'S SINGULAR INTEGRODIFFERENTIAL EQUATION

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SUMMARY

Prandtl's singular integrodifferential equation and related equations appear in problems of aerofoil and propeller theory in fluid mechanics. Here a natural interpolation formula for the approximation to the unknown function of Prandtl's equation when this is solved numerically by the direct quadrature method, based on the Gauss- and Lobatto–Chebyshev quadrature rules, is proposed. This interpolation formula is analogous to Nyström's natural interpolation formula for Fredholm integral equations of the second kind and the corresponding formula for singular integral equations. Numerical applications of the same formula are also made.

KEY WORDS Prandtl's Equation Singular Integrodifferential Equations Quadrature Method Natural Interpolation Formula Principal Value Integrals Gauss- and Lobatto–Chebyshev Quadrature Rules

INTRODUCTION

Prandtl's singular integrodifferential equation:[1,2]

$$\frac{\Gamma(x)}{B(x)}-\frac{1}{2\pi}\int_{-1}^{1}\frac{\Gamma'(t)}{t-x}\,dt=f(x), \qquad -1\leqq x\leqq 1 \tag{1}$$

where $\Gamma(x)$ is the unknown function and $B(x)$ and $f(x)$ are known functions, is a classical equation in fluid mechanics (associated with aerofoils, aircraft wings of finite span and propeller theory). The same equation is supplemented by the conditions

$$\Gamma(1)=\Gamma(-1)=0 \tag{2}$$

Derivations of (1) and discussion of its usefulness in engineering applications and its restrictions in complicated current problems of aerofoil theory (where more complicated integral equations frequently substitute for (1)) can be found in many monographs, e.g. those by Van Dyke,[3] Ashley and Landhal[4] and Karamcheti.[5] In spite of its restrictions, (1) has been used in hundreds of papers up to now and it is still in use.

Among the extensive literature on (1), we can mention a chapter in the classical monograph by Muskhelishvili,[6] where further references are reported. In this monograph, a method for reducing (1) to a regular Fredholm integral equation of the second kind is described. Unfortunately, the resulting equation is too complicated (compared to (1)) to be of practical use for the numerical solution of (1) by using the quadrature method for Fredholm integral equations of the second kind suggested by Nyström.[7] Multhopp[8] proposed

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0271–2091/84/030283–08$01.00

Received 6 August 1981
Revised 14 December 1982

a direct collocation method for the numerical solution of (1) (the classical Multhopp's method), based on the change of variables

$$t = \cos \tau, \qquad x = \cos \theta \tag{3}$$

and the application of simple properties of trigonometric functions. Multhopp's method is also reported by Kalandiya[9] and Sharfuddin.[10] Kalandiya[9,11] and Schleiff[12] considered also the convergence of Multhopp's method under appropriate conditions. On the other hand, Rösel[13] suggested an interesting alternative method of solution of (1), based on a completely new (but equivalent) form of it.

Recently, the author modified Multhopp's method so that it is not based on a change of variables (like (3)) and generalized it to singular integrodifferential equations more complicated than (1). His results are reported in a paper by him and Theocaris[14] and reduce (1) to a system of linear equations, the solution of which determines the values of the unknown function in (1) at a set of appropriately selected nodes. It is further desirable to approximate the unknown function in (1) along the whole interval $[-1, 1]$. In Multhopp's method this is made by a trigonometric series based on the solution of the aforementioned system of linear equations. This is equivalent to a Langrangian interpolation formula.

Here we suggest a natural interpolation formula for the approximation to the unknown function in (1) along the whole interval $[-1, 1]$. This formula is analogous to Nyström's natural interpolation formula for Fredholm integral equations of the second kind[7] (and is believed to become as popular as Nyström's) and is based on the error terms of the quadrature rules used for the reduction of (1) to a system of linear equations, exactly as has been recently the case for singular integral equations (with Cauchy-type kernels). A natural interpolation formula for this class of equations was also proposed by the author.[15,17]

THE INTERPOLATION FORMULA

Following the developments of Reference 14, we replace the unknown function $\Gamma(x)$ in (1) by the new unknown function $h(x)$, where

$$\Gamma(x) = w(x)h(x) \tag{4}$$

with the weight function

$$w(x) = (1 - x^2)^{-1/2} \tag{5}$$

Moreover, the conditions (2) can be written as

$$h(1) = h(-1) = 0 \tag{6}$$

Now (1) can be written as[14]

$$\frac{w(x)h(x)}{B(x)} - \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1} w(t)\frac{h(t)}{t-x}\,\mathrm{d}t = f(x), \qquad -1 \leqq x \leqq 1 \tag{7}$$

We use the Gauss- and Lobatto–Chebyshev quadrature rules (exactly as in Reference 14)

$$\int_{-1}^{1} w(t)h(t)\,\mathrm{d}t \simeq \frac{\pi}{n}\sum_{i=1}^{n} h(t_i) \tag{8}$$

$$\int_{-1}^{1} w(t)h(t)\,\mathrm{d}t \simeq \frac{\pi}{n}\sum_{k=1}^{n-1} h(y_k) \tag{9}$$

respectively (taking also into account (6)), where the nodes t_i and y_k are determined by

$$T_n(t_i)=0 \quad \text{or} \quad t_i=\cos\theta_i, \qquad \theta_i=(i-0{\cdot}5)\pi/n, \qquad i=1(1)n \tag{10}$$

$$U_{n-1}(y_k)=0 \quad \text{or} \quad y_k=\cos\theta_k^*, \qquad \theta_k^*=k\pi/n, \qquad k=1(1)(n-1) \tag{11}$$

where $T_n(x)$ and $U_n(x)$ denote the Chebyshev polynomials of degree n of the first and the second kind, respectively.

We also take into account the Gauss– and Lobatto–Chebyshev quadrature rules for the derivatives of Cauchy-type principal value integrals[14]

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1} w(t)\frac{h(t)}{t-x}\,\mathrm{d}t \simeq \frac{\pi}{n}\sum_{i=1}^{n}\frac{h(t_i)}{(t_i-x)^2}+K'_{nG}(x)h(x)+K_{nG}(x)h'(x) \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1} w(t)\frac{h(t)}{t-x}\,\mathrm{d}t \simeq \frac{\pi}{n}\sum_{k=1}^{n-1}\frac{h(y_k)}{(y_k-x)^2}+K'_{nL}(x)h(x)+K_{nL}(x)h'(x) \tag{13}$$

where[14,18]

$$K_{nG}(x)=\frac{\pi U_{n-1}(x)}{T_n(x)}, \qquad K_{nL}(x)=\frac{\pi T_n(x)}{(x^2-1)U_{n-1}(x)} \tag{14}$$

By taking into account the second of (3), we can see that

$$K_{nG}(x)=\pi\tan n\theta/\sin\theta, \qquad K_{nL}(x)=-\pi\cot n\theta/\sin\theta \tag{15}$$

and, furthermore, that

$$K_{nL}(x)-K_{nG}(x)=-\frac{2\pi}{\sin\theta\sin 2n\theta} \tag{16}$$

Next, by differentiating (15) with respect to x (taking always into account the second of (3)), we find that

$$K'_{nG}(x)=\pi\frac{\sin n\theta\cos n\theta\cos\theta-n\sin\theta}{\sin^3\theta\cos^2 n\theta}=\pi\frac{xU_{n-1}(x)T_n(x)-n}{(1-x^2)T_n^2(x)} \tag{17}$$

$$K'_{nL}(x)=-\pi\frac{\sin n\theta\cos n\theta\cos\theta+n\sin\theta}{\sin^3\theta\sin^2 n\theta}=-\pi\frac{xU_{n-1}(x)T_n(x)+n}{(1-x^2)^2U_{n-1}^2(x)} \tag{18}$$

and, furthermore, that (because of (14) and (15))

$$K'_{nG}(x)K_{nL}(x)-K'_{nL}(x)K_{nG}(x)=\frac{4\pi^2 n}{\sin^3\theta\sin 2n\theta}=\frac{4\pi^2 n}{(1-x^2)^2U_{2n-1}(x)} \tag{19}$$

By using (12) and (13) for the approximation to the integral term in (7), we find

$$\frac{w(x)h_n(x)}{B(x)}-\frac{1}{2n}\sum_{i=1}^{n}\frac{h_n(t_i)}{(t_i-x)^2}-\frac{1}{2\pi}[K'_{nG}(x)h_n(x)+K_{nG}(x)h_n'(x)]=f(x) \tag{20}$$

$$\frac{w(x)h_n(x)}{B(x)}-\frac{1}{2n}\sum_{k=1}^{n-1}\frac{h_n(y_k)}{(y_k-x)^2}-\frac{1}{2\pi}[K'_{nL}(x)h_n(x)+K_{nL}(x)h_n'(x)]=f(x) \tag{21}$$

where $h_n(x)$ denotes an approximation to $h(x)$, due to the omission of the error terms. By multiplying both sides of (20) by $K_{nL}(x)$ and both sides of (21) by $K_{nG}(x)$ and subtracting these two equations, we obtain (after a division by $K_{nL}(x)-K_{nG}(x)$)

$$\left[\frac{w(x)}{B(x)}+\frac{n}{1-x^2}\right]h_n(x)-\frac{1}{2n}\left[T_n^2(x)\sum_{i=1}^{n}\frac{h_n(t_i)}{(t_i-x)^2}+(1-x^2)U_{n-1}^2(x)\sum_{k=1}^{n-1}\frac{h_n(y_k)}{(y_k-x)^2}\right]=f(x) \quad (22)$$

where (3), (15), (16) and (19) were also taken into consideration.

By applying (22) at the nodes y_k and t_i (defined by (11) and (10), respectively) and taking into account that

$$\lim_{x\to y_k}(1-x^2)\left[\frac{U_{n-1}(x)}{y_k-x}\right]^2=\frac{n^2}{1-y_k^2}, \lim_{x\to t_i}\left[\frac{T_n(x)}{t_i-x}\right]^2=\frac{n^2}{1-t_i^2} \quad (23)$$

we find

$$\left[\frac{w(y_k)}{B(y_k)}+\frac{n}{2(1-y_k^2)}\right]h_n(y_k)-\frac{1}{2n}\sum_{i=1}^{n}\frac{h_n(t_i)}{(t_i-y_k)^2}=f(y_k), \qquad k=1(1)(n-1) \quad (24)$$

$$\left[\frac{w(t_i)}{B(t_i)}+\frac{n}{2(1-t_i^2)}\right]h_n(t_i)-\frac{1}{2n}\sum_{k=1}^{n-1}\frac{h_n(y_k)}{(t_i-y_k)^2}=f(t_i), \qquad i=1(1)n \quad (25)$$

This is a system of $(2n-1)$ linear equations in $(2n-1)$ unknowns for the determination of the approximate values of the unknown function $h(x)$ of (7) at the nodes x_j given by

$$U_{2n-1}(x_j)=0 \quad \text{or} \quad x_j=\cos[j\pi/(2n)], \qquad j=1(1)(2n-1) \quad (26)$$

For other values of x, (22) serves as a natural interpolation formula (after the numerical solution of the system of linear equations (24) and (25)), that is,

$$h_n(x)=\left[\frac{w(x)}{B(x)}+\frac{n}{1-x^2}\right]^{-1}\left\{f(x)+\frac{1}{2n}\left[T_n^2(x)\sum_{i=1}^{n}\frac{h_n(t_i)}{(t_i-x)^2}\right.\right.$$
$$\left.\left.+(1-x^2)U_{n-1}^2(x)\sum_{k=1}^{n-1}\frac{h_n(y_k)}{(y_k-x)^2}\right]\right\}, \qquad x\neq x_j, \quad j=1(1)(2n-1) \quad (27)$$

which is an approximation to the solution $h(x)$ of (7) along the whole interval $[-1, 1]$. This is the natural interpolation formula suggested here for Prandtl's singular integrodifferential equation (1).

Furhtermore, we can remark that the system of linear equations (24) and (25) coincides with the corresponding system obtained in Reference 14 by a somewhat different procedure and a somewhat different notation. Moreover, it is also directly verified ((4), (5), (10) and (11) taken into account) that the same system of linear equations coincides with the system of linear equations in the method of Multhopp[8–10] (with N in Multhopp's method taken equal to $2n-1$). In any case, the contribution of this paper is the suggestion of the natural interpolation formula (27); the system of linear equations (24) and (25) is used only for the determination of the values of $h_n(t_i)$ and $h_n(y_k)$ necessary in (27).

Now let us write (27) for the approximation $\Gamma_n(x)$ of the originally unknown function $\Gamma(x)$ in (1). Then, because of (4) and (5), we find

$$\Gamma_n(x)=\left[\frac{1}{B(x)}+nw(x)\right]^{-1}\left\{f(x)+\frac{1}{2n}\left[T_n^2(x)\sum_{i=1}^{n}\frac{\Gamma_n(t_i)}{w(t_i)(t_i-x)^2}\right.\right.$$
$$\left.\left.+(1-x^2)U_{n-1}^2(x)\sum_{k=1}^{n-1}\frac{\Gamma_n(y_k)}{w(y_k)(y_k-x)^2}\right]\right\}, \qquad x\neq x_j, \quad j=1(1)(2n-1) \quad (28)$$

where
$$\Gamma_n(t_i) = w(t_i)h_n(t_i), \qquad i = 1(1)n, \quad \Gamma_n(y_k) = w(y_k)h_n(y_k), \quad k = 1(1)(n-1) \tag{29}$$
or, equivalently (because of (10) and (11)),
$$\Gamma_n(z_j) = w(z_j)h_n(z_j), \qquad j = 1(1)(2n-1) \tag{30}$$
with
$$z_j = \cos\,[j\pi/(2n)], \qquad j = 1(1)(2n-1) \tag{31}$$
Alternatively, by using the second of (3), we can rewrite the natural interpolation formula (28) as
$$\Gamma_n^*(\theta) = \left[\frac{1}{B^*(\theta)} + \frac{n}{\sin\theta}\right]^{-1} \left\{ f^*(\theta) + \frac{1}{2n}\left[\cos^2 n\theta \sum_{i=1}^{n} \frac{\sin\theta_i \Gamma_n^*(\theta_i)}{(\cos\theta_i - \cos\theta)^2} \right.\right.$$
$$\left.\left. + \sin^2 n\theta \sum_{k=1}^{n-1} \frac{\sin\theta_k^* \Gamma_n^*(\theta_k^*)}{(\cos\theta_k^* - \cos\theta)^2}\right]\right\}, \qquad \theta = \cos^{-1} x, \quad 0 \leqq \theta \leqq \pi \tag{32}$$
where θ_i and θ_k^* are defined in (10) and (11), respectively, and
$$\Gamma_n^*(\theta) \equiv \Gamma_n(\cos\theta), \quad B^*(\theta) \equiv B(\cos\theta), \quad f^*(\theta) \equiv f(\cos\theta), \quad \cos\theta = x \tag{33}$$

On the other hand, Multhopp's interpolation formula, which is a Lagrangian interpolation (trigonometric) polynomial, has the form[8,9]
$$\tilde{\Gamma}_N^*(\theta) = \frac{1}{N+1} \sum_{j=1}^{N} (-1)^{j+1} \frac{\sin(N+1)\theta \sin\theta_j^{**}}{\cos\theta - \cos\theta_j^{**}} \Gamma_N^*(\theta_j^{**}) \tag{34}$$
where
$$\theta_j^{**} = j\pi/(N+1), \qquad j = 1(1)N \tag{35}$$
and $\tilde{\Gamma}_N^*(\theta)$ denotes the corresponding approximation to $\Gamma^*(\theta) \equiv \Gamma(\cos\theta)$. Clearly, as was already mentioned, for $N = 2n - 1$ we have
$$\Gamma_n^*(\theta_j^{**}) = \tilde{\Gamma}_N^*(\theta_j^{**}), \qquad N = 2n - 1 \tag{36}$$
because of the equivalence, under this restriction, of the system of linear equations (24) and (25) and the corresponding system of linear equations in Multhopp's method. For this reason, we have not used a tilde above Γ in the right-hand side of (34).

Of course, in general $\Gamma_n^*(\theta) \neq \tilde{\Gamma}_n^*(\theta)$ as is clear from a comparison of (32) and (34). In fact, in (32) $\Gamma_n^*(\theta)$ depends on $B^*(\theta)$ and $f^*(\theta)$ for any value of θ, whereas in (34) $\tilde{\Gamma}_N^*(\theta)$ depends only on the solution of (24) and (25), where the values of $B^*(\theta)$ and $f^*(\theta)$ are taken into account only at $\theta = \theta_j^{**}$ ($j = 1(1)(2n-1)$). This is the reason for which (32) (or, equivalently, (27) and (28)) is in general expected to give better results than (34) in practice (for $N = 2n - 1$ and the same values of $\Gamma_n^*(\theta)$). This will be verified in the numerical applications of the next section.

Finally, in the case when a regular term of the form
$$r(x) = \int_{-1}^{1} w(t)k(t, x)h(t)\,\mathrm{d}t \tag{37}$$
(where $k(t, x)$ is a regular kernel) exists in the left-hand side of (7), the above results remain still valid if we replace the right-hand side function $f(x)$ in (7) by
$$F(x) = f(x) - r(x) \tag{38}$$

and we apply the aforementioned quadrature rules to the approximation to $r(x)$ in (20), (21), etc. Further similar straightforward generalizations of the above results are quite possible.

APPLICATIONS

To test the natural interpolation formula (28) (or, equivalently, (32)) proposed in the previous section for the estimation of the unknown function $\Gamma(x)$ in (1) along the whole interval $[-1, 1]$ and to compare it with the corresponding Lagrangian interpolation formula (34) (used up to now for the same estimation), we proceeded to some numerical experiments assuming that

$$f(x) = B(x) = 1 \quad \text{or} \quad f(x) = B(x) = \cosh x \tag{39}$$

in (1) and taking $N = 2n - 1$ in (34). We repeat that the approximate values of $\Gamma(x)$, $\Gamma_n(z_j)$, used both in (28), (32) and in (34) are the same, resulting from the solution of (24) and (25) or, equivalently, form the solution of the corresponding system in Multhopp's method.[8-10] Here we test the performance of the interpolation formulae only (although the errors in $\Gamma_n(z_j)$ influence the accuracy of the numerical results).

The numerical results we obtained (for $n = 2(1)5$, $N = 2n - 1$) at the points $x_m = \cos(m\pi/11)$ $(m = 1(1)10)$ along $[-1, 1]$ are displayed in Tables I and II for the two selections of $f(x)$ and $B(x)$ in (39), respectively. From the numerical results of these tables (presented with four decimal digits) we observe their rapid convergence to the corresponding exact values (found for sufficiently large values of n, namely $n = 9$ and 10). Moreover, what is more important here, we observe (having taken into account further decimal digits whenever necessary) the superiority of the natural interpolation formula to the Lagrangian one in most cases. In fact, the natural interpolation formula is seen to be superior (giving smaller absolute errors) to the Lagrangian interpolation formula in seventeen (in both Tables I and II) out of twenty cases. The Lagrangian interpolation formula was found to be superior to the natural interpolation formula only in the remaining three cases (in both Tables I and II), namely for (i) $n = 2$ and $x_m = \pm 0{\cdot}654861$, (ii) $n = 3$ and $x_m = \pm 0{\cdot}841254$, and (iii) $n = 4$ and $x_m = \pm 0{\cdot}142315$. For $n = 5$ the natural interpolation formula was seen to be always superior to

Table I. Numerical results for the unknown function $\Gamma(x)$ in (1) (compared with the exact values of the same function) for $f(x) = B(x) = 1$ at the points $x_m = \cos(m\pi/11)$ $(m = 1(1)10)$. These results were obtained by the Lagrangian interpolation formula (34), $\tilde{\Gamma}$, and by the natural interpolation formula (28) (or (32)), Γ, for $n = 2(1)5$ and $N = 2n - 1$

θ_m	$\pi/11$ $10\pi/11$		$2\pi/11$ $9\pi/11$		$3\pi/11$ $8\pi/11$		$4\pi/11$ $7\pi/11$		$5\pi/11$ $6\pi/11$	
x_m	$\pm 0{\cdot}959493$		$\pm 0{\cdot}841254$		$\pm 0{\cdot}654861$		$\pm 0{\cdot}415415$		$\pm 0{\cdot}142315$	
n	$\tilde{\Gamma}$	Γ	$\tilde{\Gamma}$	Γ	$\tilde{\Gamma}$	Γ	$\tilde{\Gamma}$	Γ	$\tilde{\Gamma}$	Γ
2	0·2445	0·2689	0·4479	0·4592	0·5872	0·5848	0·6635	0·6592	0·6941	0·6934
3	0·2611	0·2675	0·4638	0·4629	0·5909	0·5893	0·6607	0·6611	0·6927	0·6929
4	0·2662	0·2678	0·4652	0·4643	0·5894	0·5897	0·6609	0·6610	0·6932	0·6932
5	0·2679	0·2681	0·4648	0·4646	0·5895	0·5896	0·6612	0·6611	0·6932	0·6932
Exact values	0·2686		0·4646		0·5897		0·6611		0·6932	

Table II. Similar numerical results to those of Table I, but for $f(x) = B(x) = \cosh x$

θ_m	$\pi/11$ $10\pi/11$		$2\pi/11$ $9\pi/11$		$3\pi/11$ $8\pi/11$		$4\pi/11$ $7\pi/11$		$5\pi/11$ $6\pi/11$	
x_m	$\pm 0{\cdot}959493$		$\pm 0{\cdot}841254$		$\pm 0{\cdot}654861$		$\pm 0{\cdot}415415$		$0{\cdot}142315$	
n	$\tilde{\Gamma}$	Γ	$\tilde{\Gamma}$	Γ	$\tilde{\Gamma}$	Γ	$\tilde{\Gamma}$	Γ	$\tilde{\Gamma}$	Γ
2	0·3649	0·3929	0·6337	0·6481	0·7640	0·7606	0·7848	0·7781	0·7668	0·7656
3	0·3861	0·3917	0·6533	0·6525	0·7674	0·7661	0·7798	0·7802	0·7642	0·7644
4	0·3906	0·3921	0·6546	0·6538	0·7662	0·7664	0·7801	0·7801	0·7647	0·7647
5	0·3922	0·3924	0·6543	0·6541	0·7663	0·7664	0·7803	0·7802	0·7647	0·7647
Exact values	0·3929		0·6542		0·7664		0·7803		0·7647	

the Lagrangian one. Moreover, as far as the maximum absolute error observed in the numerical results of Table I (in all twenty numerical values presented there) is concerned, it is $\tilde{\varepsilon} = 0{\cdot}0241$ for the Lagrangian interpolation formula and $\varepsilon = 0{\cdot}0054$ for the natural interpolation formula. In Table II the corresponding values are: $\tilde{\varepsilon} = 0{\cdot}0280$ and $\varepsilon = 0{\cdot}0061$, respectively.

CONCLUSIONS

The previous results show that in most cases the natural interpolation formula for Prandtl's singular integrodifferential equation gives better numerical results than the corresponding Lagrangian interpolation formula used up to now. Of course, in some particular cases the contrary takes place (as observed previously) exactly as the trapezoidal quadrature rule for the estimation of integrals sometimes gives better numerical results than the corresponding Gaussian quadrature rule with the same number of nodes. It is believed that for Prandtl's singular integrodifferential equation the natural interpolation formula will gradually substitute for the Lagrangian interpolation formula as has been the case for Fredholm integral equations for over fifty years[7] and for Cauchy type singular integral equations recently.[15–17]

ACKNOWLEDGEMENT

The results reported here belong to a research project supported by the National Hellenic Research Foundation. The author gratefully acknowledges the financial support of this Foundation.

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